Supersonic Ducts at the Angle of Attack

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1. Introduction

The solution of the problem of the gas flow about a body at the angle of attack consists of the integration of three-dimensional non-linear system of gasdynamic equations and may be obtained efficiently only by numerical methods on electronic digital computers. Numerical calculations give not only the total aerodynamic forces acting on the flying body but the flow fields as well. In the numerical calculations it is not difficult to take into account the physical–chemical processes taking place in the gas stream at high temperatures.

For the calculation of three-dimensional supersonic flow about a body it is possible to use different numerical methods such as the finite difference method, the method of integral relations or the method of characteristics. A number of numerical schemes of the method of characteristics was proposed by different authors.

For the analysis of three-dimensional supersonic gas flow about a body the method of characteristics has certain merits since it possesses a larger mathematical rigour and accurately takes into account the propagation of disturbances — the last property is of special importance for investigation of the flow about a body with discontinuity of contour and when secondary shock waves arise inside the flow field. Also in the method of characteristics streamlines are determined along which physical–chemical processes take place; this simplifies the calculation of these phenomena. At the same time some limitations are inherent in the method of characteristics in its pure form. These limitations are connected with laborious computations and complicated shapes of characteristic surfaces.

Therefore the application of so-called semi-characteristic schemes is of interest for the calculation of three-dimensional supersonic flow about smooth bodies. In such schemes one of the independent variables is eliminated by means of approximation and the problem is essentially reduced to the solution of a system of equations in two variables but for the larger number of unknown functions. This approximating system may be integrated numerically by the two-dimensional method of characteristics.

In comparison with the usual method of characteristics it requires less computation and the programme logic is simpler.

The numerical scheme of such a type in which cylindrical co-ordinates are considered and trigonometrical interpolations in the angular variable ψ are used was developed by O. N. Katskova and P. I. Chushkin(1). This scheme has second-order accuracy. Here the calculations are carried out along strips perpendicular to the axis of the body. In each meridian plane the characteristics are projected upstream in the direction of the previous strip and the nodes of network are placed in the points fixed beforehand.

A semi-characteristic scheme was applied also by G. Moretti(2) for the calculation of supersonic flow about bodies. However this scheme, in comparison with the above(1), has some shortcomings. It has only the first-order accuracy (the characteristics are projected downstream), only linear approximations in ψ are employed and the straightening out of the considered region is not carried out, which creates certain difficulties at calculation near the boundaries.† R. Sauer(3) also described one semi-characteristic scheme, but no results of calculations were presented.

In this report our numerical semi-characteristic scheme is applied for the calculation of the external supersonic flow about a duct at the angle of attack with the attached shock wave.

2. Numerical Method

Let us describe briefly the method applied for calculation of the supersonic steady flow about a body placed at the angle of attack α in the uniform gas stream with velocity $\mathbf{V}_\infty$ and Mach number M_∞. For simplicity we confine ourselves to the case of the perfect gas with the constant specific heat ratio γ.

Take the cylindrical system of co-ordinates x, r, ψ connected with the body (Fig. 1). The body is supposed to have the plane of symmetry in which the velocity vector $\mathbf{V}_\infty$ lies; let the plane $\psi=0$ corresponds to the windward side of the body and the plane $\psi=\pi$ to the leeward side. The flow must be calculated in the region bounded by the given body surface $r=r_B(x,\psi)$ and the unknown surface of the shock wave $r=r_S(x,\psi)$ for $0\leqq\psi\leqq\pi$.

The system of gasdynamic equations is taken in the following form

† Recently, G. Moretti in a brief note (*AIAA Journal*, 1966, **4**, No. 9, 1695) has communicated certain improvements in his scheme.

$$\rho a^2 \mathbf{V}\mathbf{V} + \mathbf{V}\nabla p = 0, \quad \rho(\mathbf{V}\nabla)\mathbf{V} + \nabla p = 0, \quad \mathbf{V}\nabla s = 0 \tag{1}$$

where ∇, p, s, ρ, a denote velocity vector, pressure, entropy, density and velocity of sound respectively; as is known the last two functions are

$$\rho = p^{1/\gamma} \exp\left(\frac{1-\gamma}{\gamma} s\right), \quad a^2 = \gamma \frac{p}{\rho}$$

Here all the functions are assumed to be dimensionless, with some length (for ducts it is the entrance radius), the free stream velocity V_∞ and the ambient density ρ_∞ being used as the reference quantities.

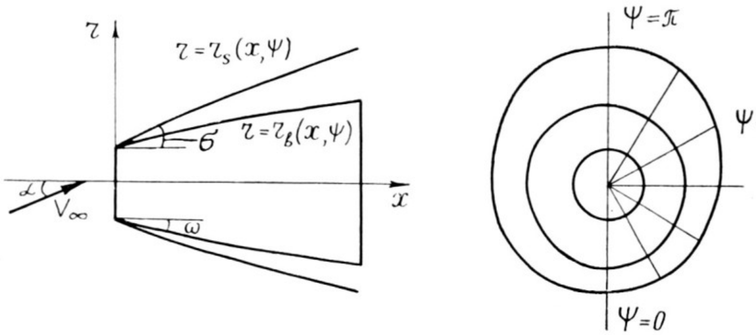

FIG. 1 — Flow about a duct at the angle of attack

By introducing instead of r the new independent variable ξ

$$\xi = \frac{r - r_B(x, \psi)}{r_S(x, \psi) - r_B(x, \psi)}$$

the region of integration on the plane $x = \text{const}$ is transformed into the circular annulus. Then the system (1) in the variables x, ξ, ψ will contain the derivatives of five basic unknown functions — velocity components u, v, w along the cylindrical co-ordinates axes; pressure p; entropy s.

In addition the following functions λ and μ are included in this system

$$\lambda = -\xi(r'_{Sx} - r'_{Bx}) - r'_{Bx}, \quad \mu = -\frac{1}{r}[\xi(r'_{S\psi} - r'_{B\psi}) + r'_{B\psi}]$$

where the prime and the subscripts denote the corresponding derivatives.

From this transformed three-dimensional system the independent variable ψ will be eliminated by means of approximations of all the functions with trigonometric polynomials in ψ, for which the meridian planes $\psi = \psi_k = k\pi/l$ $(k = 0, 1, \ldots, l)$ are served as the nodes of interpolation. In this case the even functions $f=u, v, p, s$ and the odd function w will be approximated respectively by the expressions

$$f(x,\xi,\psi)=\sum_{k=0}^{l} b_k(x,\xi)\cos k\psi, \quad w(x,\xi,\psi)=\sum_{k=1}^{l-1} a_k(x,\xi)\sin k\psi \tag{2}$$

As a result we obtain the approximating system of two-dimensional equations in x and ξ with respect to the values of the basic unknown functions on all the meridian interpolational planes $\psi=\psi_k$. It is found that for the condition

$$u^2+(v+\mu w)^2/(1+\mu^2) > a^2$$

this approximating system will be hyperbolic possessing two families of real characteristics and one family of the lines which by analogy with the axi-symmetric case may be called conditionally the 'streamlines'. For the convenience of calculations on digital computers it is advisable to introduce two new functions

$$\zeta=\frac{1}{u}(v+\mu w) \qquad \text{and} \qquad \beta=\left[\frac{u^2\zeta^2}{a^2}+(1+\mu^2)\left(\frac{u^2}{a^2}-1\right)\right]^{1/2}$$

In this case the differentional characteristic equations and the compatibility relations can be written as follows

$$\begin{gathered}\frac{\mathrm{d}\xi}{\mathrm{d}x}=\frac{1}{\varepsilon}\left(\lambda+\frac{u^2\zeta\pm a^2\beta}{u^2-a^2}\right)\\ \mathrm{d}\zeta\pm\frac{\beta}{\rho u^2}\mathrm{d}p+\frac{1}{u^2}\left[\Phi_1\frac{ua^2(\zeta\pm\beta)}{u^2-a^2}-\Phi_2\frac{u^2\zeta\pm a^2\beta}{u^2-a^2}+\Phi_3+\mu\Phi_4-uw\frac{\mathrm{d}\mu}{\mathrm{d}x}\right]=0\end{gathered} \tag{3}$$

where the different signs correspond to the characteristics of the different families.

For the 'streamlines' we have the following differential equation and relations

$$\left.\begin{aligned}\frac{\mathrm{d}\xi}{\mathrm{d}x}&=\frac{1}{\varepsilon}(\lambda+\zeta)\\ \mu\,\mathrm{d}v-\mathrm{d}w+\frac{1}{u}(\mu\Phi_3-\Phi_4)\,\mathrm{d}x&=0\\ \mathrm{d}u+\zeta\,\mathrm{d}v+\frac{1}{\rho u}\mathrm{d}p+\frac{1}{u}(\Phi_2+\zeta\Phi_3)\,\mathrm{d}x&=0\\ \mathrm{d}s+\frac{\Phi_5}{u}&=0\end{aligned}\right\} \tag{4}$$

In equations (3)–(4)

$$\varepsilon = r_S(x,\psi) - r_B(x,\psi)$$

$$\Phi_1 = \frac{1}{r}\left(\frac{\partial w}{\partial \psi} + \frac{w}{\rho a^2}\frac{\partial p}{\partial \psi} + v\right)$$

$$\Phi_2 = \frac{w}{r}\frac{\partial u}{\partial \psi}$$

$$\Phi_3 = \frac{w}{r}\left(\frac{\partial v}{\partial \psi} - w\right)$$

$$\Phi_4 = \frac{1}{r}\left(w\frac{\partial w}{\partial \psi} + \frac{1}{\rho}\frac{\partial p}{\partial \psi} + vw\right)$$

$$\Phi_5 = \frac{w}{r}\frac{\partial s}{\partial \psi}$$

The equations (3) and (4) are represented in the finite-difference form by the scheme of second-order accuracy. The solution is found step by step on successive planes $x=\text{const}$ where the nodal points are chosen at fixed values $\xi=\text{const}$ equal for each of the $l+1$ considered rays $\psi=\text{const}$. The characteristics and the 'streamlines' are projected from these nodal points to the intersection with the previous plane $x=\text{const}$. The necessary values of functions in these intersection points are determined by interpolation. The flow functions in the nodes of networks are found by iterations from the finite-difference system.

While solving the problem it is necessary to calculate nodal points inside the flow field, on the shock wave ($\xi=1$) and on the body surface ($\xi=0$). The gasdynamic relations for the strong discontinuity are used additionally for the calculation of the shock wave points. The condition of vanishing of normal velocity is applied for the calculation of the body surface points. The detailed numerical procedures are given in ref. 1 for all these cases. In that paper the present numerical method is developed in the general form for the equilibrium flow of real gas.

The choice of $l+1$ number of interpolational planes $\psi=\psi_k$ and number of nodal points in each of such planes depend on the required accuracy of solution. The step size Δx is determined by the numerical stability condition.

3. Some Numerical Results

The described numerical method was used for the calculation of supersonic flows about various bodies with blunt or sharp nose parts. Here we shall represent some numerical results for the external flow about ducts at the angle of attack in the supersonic stream of perfect gas. We considered the case when the shock wave is attached to the leading edge of the duct and when therefore the flow about external surface of the inlet does not depend on the flow inside the duct (Fig. 1).

Before the calculation of the entire flow field it is necessary to obtain the initial data at the entrance of the duct at $x=0$. The inclination angle of the shock wave σ to the x-axis at the leading edge of the body of revolution is determinated only by the x and r velocity components of the incident stream

$$u_\infty = V_\infty \cos\alpha, \quad v_\infty = -V_\infty \sin\alpha\cos\psi$$

While the θ velocity component $w_\infty = V_\infty \sin\alpha\sin\psi$ does not affect the quantity σ and remains continuous across the shock wave. Therefore it should be considered an effective angle of attack $\bar{\alpha}$ and an effective Mach number M_∞ which are expressed as

$$\tan\bar{\alpha} = \tan\alpha\cos\psi, \quad \overline{M}_\infty = M_\infty \frac{\cos\alpha}{\cos\bar{\alpha}}$$

Hence for the given inclination angle of duct surface at the leading edge ω

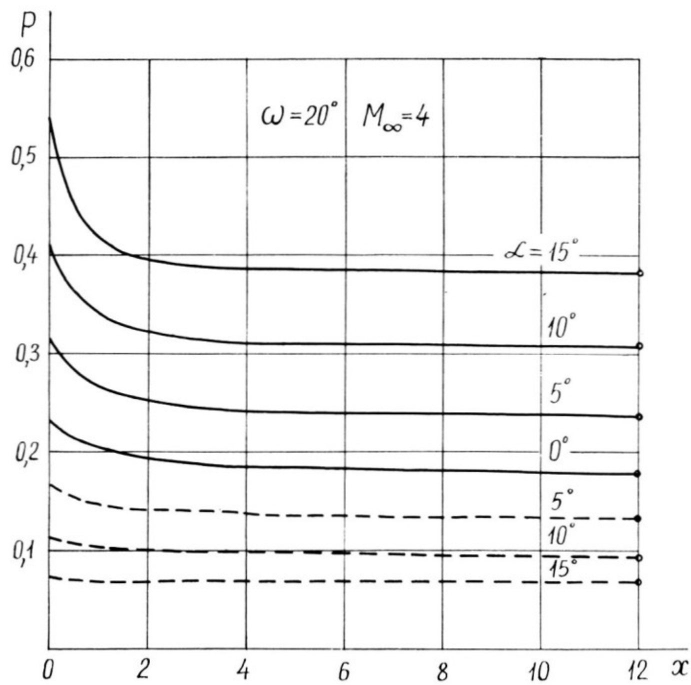

FIG. 2 — Pressure distribution along the duct with $\omega=20°$ for $M_\infty=4$ at various angles of attack α

the value σ will be found as for the corresponding wedge in the stream with the Mach number $\overline{M}_\infty$ and the angle of attack $\bar{\alpha}$. Then the inclination angle of the shock wave σ will be calculated by iterations from the next trigonometric equation

$$\frac{1}{\tan(\omega+\bar{\alpha})} = \left(\frac{\gamma+1}{2}\frac{\overline{M}_\infty^2}{\overline{M}_\infty^2 \sin^2(\sigma+\bar{\alpha})-1}\right)\tan(\sigma+\bar{\alpha})$$

Further it is not difficult to find the values of velocity components u, v, w, pressure p and entropy s at the leading edge of the body behind the shock wave on each meridian plane $\psi=\psi_k$. The start of the solution is carried out within small step Δx with these initial data assuming all the parameters to be constant as for the flow about a wedge.

The calculations were accomplished for the supersonic flow about the ducts having the configuration of a frustum of a cone with semi-angle ω. The series of variants was calculated for different values of duct angle ω, free-stream Mach number M_∞ and angle of attack α. (Only the cases when $\alpha<\omega$ were considered, that is the rarefaction wave does not take place at the leading edge.) The specific heat ratio γ was taken as $\gamma=1{\cdot}4$.

In Figs. 2–8 some results of these calculations are presented; they were obtained for 5 meridian planes and 25 nodal points on each ray.

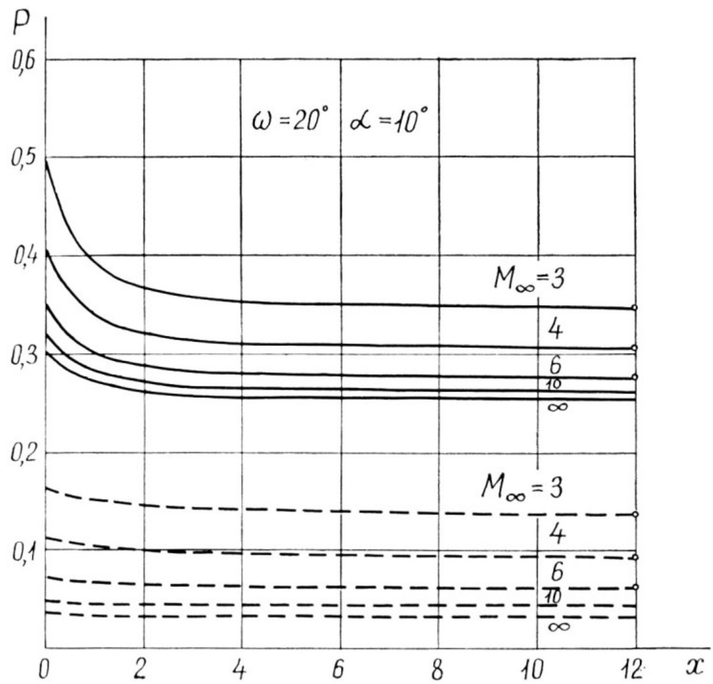

FIG. 3 — Pressure distribution along the duct with $\omega=20°$ at the angle of attack $\alpha=10°$ for various free-stream Mach numbers M_∞

The pressure distribution $p=p(x)$ along the surface of the body in the gas stream with Mach number $M_\infty=4$ is plotted in Fig. 2 for different angles of attack. The solid lines correspond to the windward side $\psi=0$ and the dotted lines correspond to the leeward side $\psi=\pi$. In the intermediate planes $\psi=\text{const}$ the curves $p=p(x)$ have analogous forms. The small circles here and further denote the appropriate results for the flow about the sharp cone with the same semi-angle ω, calculated by the finite-difference method of ref. 4. As is obvious for the flow about the duct at the angle of attack the pressure approaches rapidly to the asymptotic value corresponding to the sharp cone.

The similar graph of the pressure distribution is given in Fig. 3 for the duct with $\omega=20°$, $\alpha=10°$ and for the various values of free-stream Mach number M_∞. In Fig. 4 the shock wave traces on the planes $\psi=0$ and $\psi=\pi$

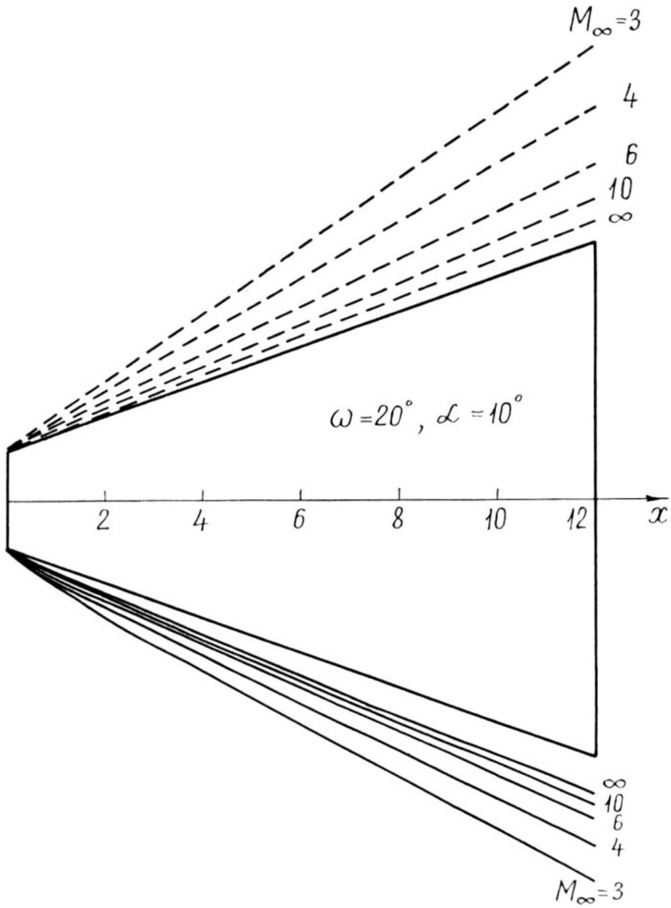

FIG. 4 — The shock waves in the plane of symmetry for the duct $\omega=20°$ at $\alpha=10°$ and various Mach numbers M_∞

are shown for the same case. At large distances the shock wave inclinations tend also to the corresponding inclinations for the sharp cone. It is interesting that for the hypersonic Mach numbers the shock layer thickness on the windward side is less than that on the leeward side.

The variation of the pressure across the shock layer between the body surface $\xi=0$ and the shock wave $\xi=1$ is given in Fig. 5. These results are

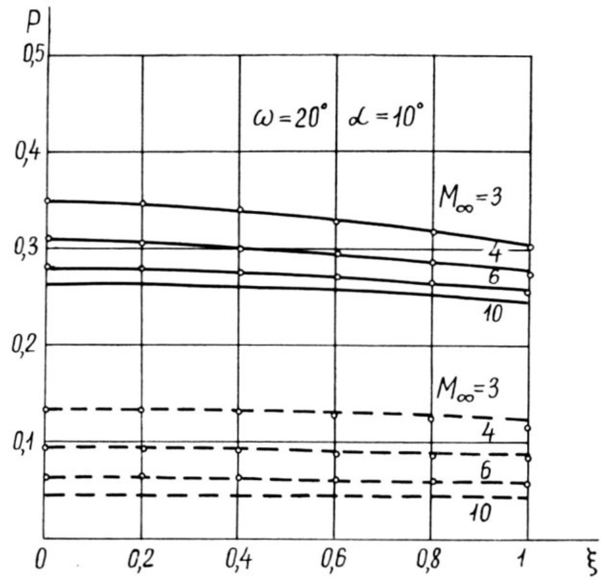

FIG. 5 — Variation of pressure across the shock layer for the duct $\omega=20°$ at $\alpha=10°$ and various Mach numbers M_∞

obtained for the duct with $\omega=20°$ at $\alpha=10°$ and various values of M_∞. The pressure profiles are shown for sufficiently large distance $x=14$ where these curves are already close to the corresponding dependences for the sharp cone. It can be seen from this graph that the more rapid approach takes place for the greater Mach number M_∞ and on the windward side.

The behaviour of entropy s on the external surface of a duct is interesting at different values of x. In contrast to the sharp cone the external surface of the duct at the angle of attack is not the constant entropy surface. Here the lines of constant entropy represent some curves and only two lines $s=\text{const}$ lying in the plane of symmetry of the flow will be straight. Figure 6 shows the entropy dependence $s=s(\psi)$ on the body surface for several planes $x=\text{const}$ in the case $\omega=20°$, $M_\infty=4$ and $\alpha=10°$.

The pressure distribution against angular variable ψ is plotted on Fig. 7. The shape of the shock wave $r=r_S(\psi)$ in the planes $x=\text{const}$ are drawn in Fig. 8 for $\omega=10°$, $M_\infty=4$ and $\alpha=5°$. The curves shown in these two graphs are

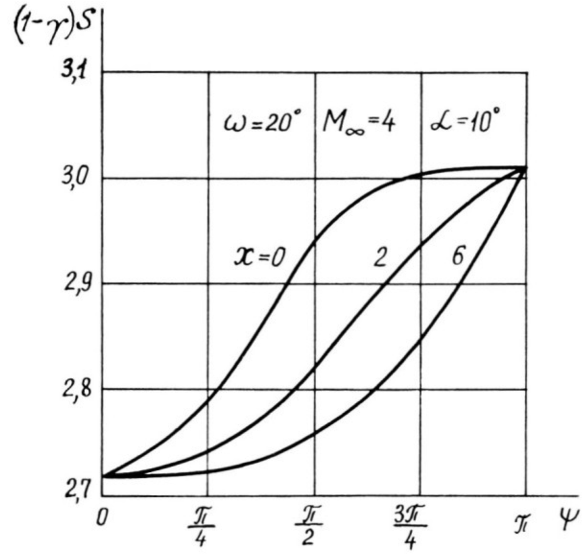

FIG. 6 — The entropy dependence on the body surface for $\omega=20^\circ$, $M_\infty=4$ and $\alpha=10^\circ$

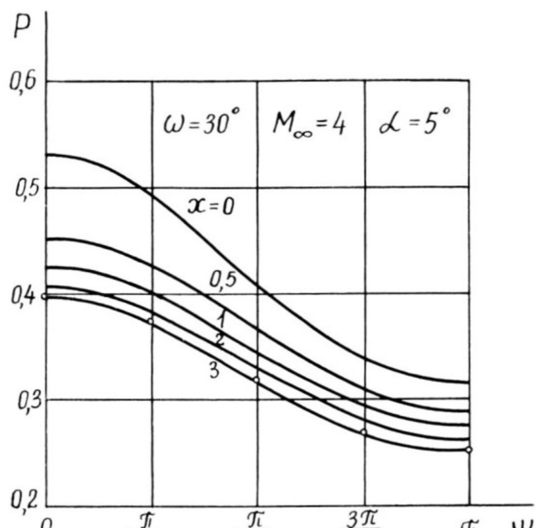

FIG. 7 — Pressure distribution on the body surface for $\omega=30^\circ$, $M_\infty=4$ and $\alpha=5^\circ$

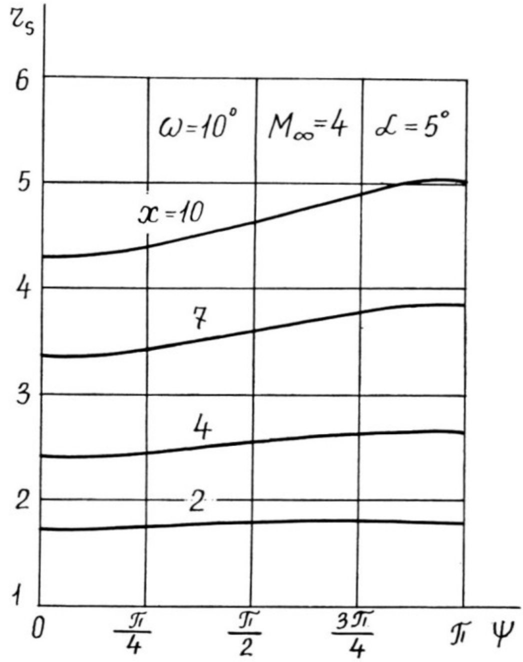

FIG. 8 — The shape of shock wave in the planes $x = \text{const}$ for duct $\omega = 10°$ at $M_\infty = 4$ and $\alpha = 5°$

sufficiently smooth and this is important for the approximation with trigonometric polynomials (2) used in the present numerical method.

For the calculation of the external supersonic flow about ducts at the angle of attack it is possible to compute not only the basic gasdynamic functions but also other different interesting quantities, e.g. the temperature. Furthermore it is easy to calculate the coefficients of total aerodynamic forces acting on the external surface of the duct (drag and lift coefficients, moment coefficient and so on). Using the method developed in ref. 1 it is possible also to compute the flow about the ducts in the thermodynamic equilibrium gas stream.

In conclusion we would note that at present we calculate the large series of supersonic flows about ducts at the angle of attack in the stream of perfect gas. These results are supposed to be included in the corresponding tables.

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